Lecture 6

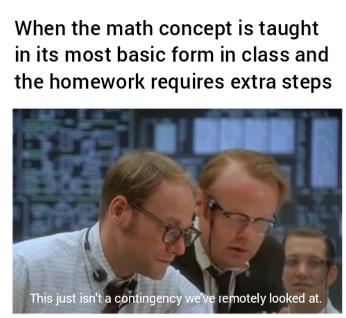
Conditional Probability and Independence

- In this lecture, we will talk about conditional probability.
- The general idea is that the probability of some given event occurring may be different, modulo other information.

Look! A tiny message!

- Example: suppose we have a well-shuffled deck of cards, and suppose you have a friend and that friend takes the top card. If you take the second card, what is the probability that it's an ace?
  - ↳ if you know that your friend drew an ace, then the probability is $3/51$.
  - ↳ if not, then the probability is $4/51$.

This is an example of Conditional Probability!



We write $P(E|F)$ to mean "the probability that event E occurs assuming that event F has occured."

In general for $P(F) \neq 0$,

$$P(E|F) = \frac{P(EF)}{P(F)}$$

<u>Ex:</u> Suppose we are rolling two die, one after the other. What is the probability that the sum of the rolls is greater than 7 given that the first roll is a 3?

<u>Sol'n:</u> Let E = "the sum is greater than 7."
F = "the first roll is a 3."

$P(F) = 1/6$, since each out come is equally likely and there are 6 sides.

$P(EF) \rightsquigarrow$ for two rolls, there are 36 possible out comes, 6 of which start with three:
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)
Only (3,5) and (3,6) sum to greater than 7, so $P(EF) = 2/36$.

Hence: $P(E|F) = 2/36 / 1/6 = \frac{12}{36} = 1/3$.

But note $1/3 = 2/6$

The sample space is "reduced" to

$(3,1), (3,2), (3,3), (3,4), \underbrace{(3,5), (3,6)}_{2 \text{ (out of } 6)}$

i.e. is restricted to F

since F has occured, F becomes our new sample space the "reduced sample space".

Example: In bridge, 52 cards are dealt out equally to four players called East, West, North, and South.
If north and south have a total of 8 spades what is the probability that East has three of the remaining 5 spades?

Soln: We consider the reduced sample space:
Since N and S have been dealt 26 cards together, among the other 26, there are $\binom{26}{13}$ possible hands for E.

Among these hands, there are

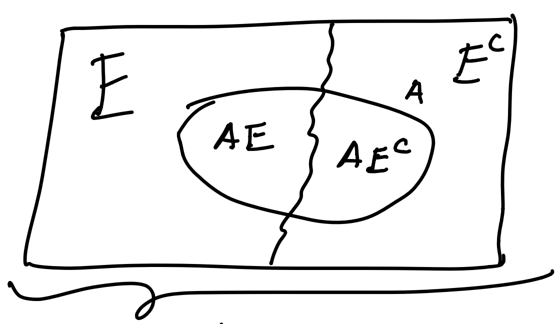
$$\binom{5}{3}\binom{21}{10} \text{ many hands with 3 spades.}$$

5 spades, choose 3 — rest of hand

So the prob. is $$\frac{\binom{5}{3}\binom{21}{10}}{\binom{26}{13}} \approx 0.339.$$

Note that
$$P(E|F) = \frac{P(EF)}{P(F)}$$
so
$$P(EF) = P(E|F)P(F).$$ ← "multiplication rule"

This can be used to give the following convenient tool:



From the picture: $$P(A) = P(AE) + P(AE^c)$$

So, by the multiplication rule:

$$P(A) = P(AE) + P(AE^c)$$

$$\underbrace{P(A) = P(A|E)P(E) + P(A|E^c)P(E^c)}$$

"Law of total probability".

Ex: Given a well shuffled deck of cards, what is the probability that the second card is an ace? Let

E = the first card is an ace.

A = the second card is an ace.

$$P(A) = P(\cancel{A}|E)P(E) + P(A|E^c)P(E^c)$$

$$P(A|E) = 3/51 \qquad P(A|E^c) = 4/51$$

$$P(E) = 4/52 \qquad P(E^c) = 48/52$$

More generally, if $\{E_i\}_i$ is a partition of S (so $\bigcup_i E_i = S$ and $E_i \cap E_j = \emptyset$ if $i \neq j$) then

$$\boxed{P(A) = \sum_i P(A|E_i)P(E_i)}$$ "Law of Total probability"

Generalized Multiplication Rule

$$P(E_1E_2\ldots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\ldots P(E_n|E_1\ldots E_{n-1})$$

Ex: A deck of cards is randomly shuffled into 4 piles. What is the probability that each pile has exactly one ace?

Sol: We consider the following events:

E_1 = "ace of spades is in any of the piles"

E_2 = "ace of spades and ace of hearts are in different piles"

E_3 = "the aces of spades, hearts, diamonds are all in different piles"

E_4 = "all aces in different piles

We want $P(E_1E_2E_3E_4)$ and so

$$P(E_1E_2E_3E_4) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3).$$

$P(E_1) = 1$ since the ace of spades must end up somewhere.

$P(E_2|E_1)$: Consider the pile containing the ace of spades. The probability that the ace of hearts is one of the 12 other cards is $12/51$.

So $P(E_2 | E_1) = 1 - \frac{12}{51} = \frac{39}{51}$.

$P(E_3 | E_1 E_2)$: Consider the two piles containing the aces of hearts and spades. The probability that ace of diamonds is in one of the two piles is $24/50$, so

$$P(E_3 | E_1 E_2) = 1 - \frac{24}{50} = \frac{26}{50}.$$

$P(E_4 | E_1 E_2 E_3)$: Now we have three piles each with an ace. The probability that the last ace is in one of the piles is $\frac{36}{49}$, so

$$P(E_4 | E_1 E_2 E_3) = 1 - \frac{36}{49} = 13/49$$

So $P(E_1 E_2 E_3 E_4) = 1 \cdot \frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49}$

$= 0.105.$